Multi-dimensional Limiting Process and Maximum Principle for the Computations of Hyperbolic Conservation Laws on Unstructured Grids

Jin Seok PARK[1] and Chongam KIM[1, 2]

1) *School of Mechanical Aerospace Eng., Seoul National University, Seoul 151-742, KOREA*
2) *Institute of Advanced Aerospace Technology, School of Mechanical and Aerospace Eng., Seoul National University, Seoul, 151-742, KOREA*

Corresponding Author: Chongam KIM, chongam@snu.ac.kr

ABSTRACT

The present paper presents an efficient and accurate limiting strategy for the multi-dimensional hyperbolic conservation laws on unstructured grids within the framework of finite volume method. The basic idea is to control the distribution of both cell-centered and cell-vertex physical properties to mimic a multi-dimensional nature of flow physics, which can be formulated as so called MLP condition. Mathematically, this condition satisfies the maximum principle which is a complementary condition ensuring monotonicity. Various numerical results show that the MLP is quite effective and accurate in preventing unwanted oscillations and capturing multi-dimensional flow features.

INTRODUCTION

Robust high resolution scheme is essential to solve large scale problems accurately and efficiently. With a high-order accurate reconstruction scheme, it is feasible to capture complex flow structure with adequate computational cost. At the same time, it entails non-physical oscillations near discontinuities such as shock waves. Spurious oscillations may lead to wrong solution as well as serious convergence problem. Therefore, a robust and accurate oscillation control strategy should be incorporated into a higher-order interpolation scheme. However, most oscillation-free schemes including TVD and ENO-type schemes are mainly based on the mathematical analysis of one-dimensional convection equation, and applied to systems of equations with the help of some linearization step. Though this approach may work successfully in many cases, it is often insufficient or almost impossible to control oscillation near shock discontinuity in multi-dimensional flow.

In order to find out a suitable criterion for oscillation control in multiple dimensions, the one-dimensional monotonic condition was extended to multi-dimensional flow situations and the multi-dimensional limiting process (MLP) was successfully developed. From the series of researches, it has been clearly demonstrated that the MLP limiting strategy possesses favorable characteristics, such as enhanced accuracy and convergence behaviors in inviscid and viscous computations on structured grids [1, 2]. Recently this strategy is also extended on two-dimensional triangular grids [3]. It was observed that MLP on unstructured grids is quite effective to control multi-dimensional oscillations as well as accurate in capturing multi-dimensional flow features. In this work, we explore monotonicity of MLP on 2-D and 3-D unstructured grids.

MULTI-DIMENSIONAL LIMITING PROCESS

In order to maintain multi-dimensional monotonicity, the present limiting strategy exploits the MLP condition, which is an extension of the one-dimensional monotonic condition. The basic idea of MLP condition is to control the distribution of both cell-centered and cell-vertex physical properties to mimic a multi-dimensional nature of flow physics. Especially, we focus on hypothesis that well-controlled vertex values at interpolation stage make it possible to produce monotonic distribution of cell-averaged values. Based on this idea, the vertex values are required to satisfy the following MLP condition (Eq. (1)).

$$\overline{q}_{neighbor}^{\min} \le q_{vtx} \le \overline{q}_{neighbor}^{\min}, \tag{1}$$

where q_{vtx} is the value at vertex and $\left(\overline{q}_{neighbor}^{\min}, \overline{q}_{neighbor}^{\min}\right)$ are the minimum and maximum values among the neighboring cell-averaged values sharing the vertex. Above MLP condition can be implemented regardless of grid topology basically. In order to enhance efficiency and accuracy, however, how to approximate the vertex value q_{vtx} can be suited to structured or unstructured grid.

On structured grids, a physical property at vertex can be efficiently estimated by summing the one dimensional variation along each coordinate direction. Thus, the MLP limiting can be readily implemented within the TVD-MUSCL framework by adopting the variable limiting region. On the other hand, there is no explicit reference direction on unstructured grid system, and directional variation cannot be readily obtained. To cope with the multi-dimensional nature, the interpolation stage simply starts from the MUSCL-type framework on unstructured grids.

$$q_j(\mathbf{x}) = \overline{q}_j + \phi \nabla \overline{q}_j \cdot \mathbf{r}, \tag{2}$$

In this framework, the maximum or minimum value occur on the vertices of the cell, and thus the value at each vertex should be limited by the MLP condition. The MLP slope limiter is introduced to ensure monotonicity. With lengthy derivations, the final form of MLP slope limiter can be written as follows.

$$\phi_{MLP} = \min\begin{cases} \Phi\left(r_i^{\max}\right) & if\ \nabla q \cdot \mathbf{r}_{v_i} > 0 \\ \Phi\left(r_i^{\min}\right) & if\ \nabla q \cdot \mathbf{r}_{v_i} < 0 \\ 1 & if\ q_i = q_A \end{cases} \tag{3}$$

where $r_i^{\min/\max} = (\hat{q}_{v_i,j}^{\min/\max} - \overline{q}_j) \big/ \nabla q \cdot \mathbf{r}_{v_i}$. For monotonicity, Φ should be in the range of $0 \le \Phi(r) \le \min(1, r)$.

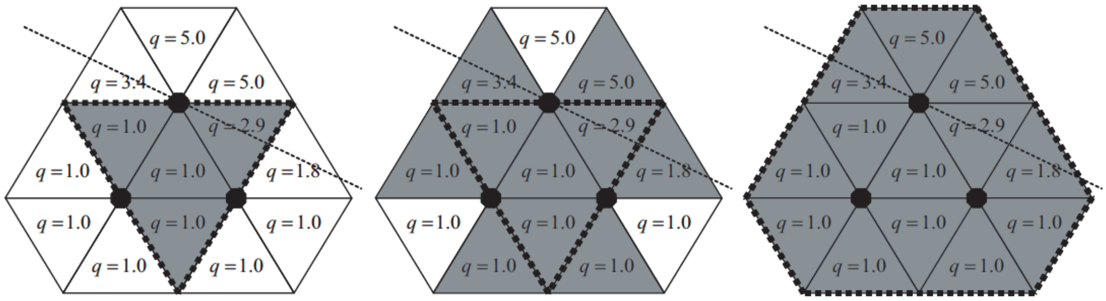
MONOTONICITY IN MULTIPLE DIMENSIONS AND MAXIMUM PRINCIPLE

The effectiveness of the MLP condition is supported by the maximum principle, which is a complementary condition ensuring the monotonicity on multiple dimensions. For the convenience of illustration, this feature is proved on a scalar conservation laws and the result is summarized in the following theorem.

Theorem. For a fully discrete finite volume scheme of hyperbolic conservation laws with a Lipschitz continuous flux function, if the linear reconstruction satisfies the MLP condition under a proper CFL restriction, then the scheme satisfies the maximum principle, i.e.,

$$\overline{q}_{j,neighbor}^{\min,n} \le \overline{q}_j^{n+1} \le \overline{q}_{j,neighbor}^{\max,n}. \tag{4}$$

$\overline{q}_{j,neighbor}^{\min,n}$ and $\overline{q}_{j,neighbor}^{\max,n}$ are the minimum and maximum cell-averaged values among the neighborhood of the cell T_j, which shares at least a common point with the cell T_j. Detail proof will be given in the presentation. Other limiters for unstructured grids, such as Barth limiter, LCD and MLG limiters, also satisfy the maximum principle. The essential difference is the stencil involved in limiting and the maximum principle (see Fig.1). Since the permissible limiting range of these limiters essentially comes from the Spekreijse's monotonic condition, only the neighboring cells which share one of the edges of the updated cell are considered. Thus, they may have drawbacks in capturing multi-dimensional flow physics. On the other hand, the MLP condition fully exploits all of the cell-averaged values sharing vertices as well as edges. More importantly, the stencils involved in limiting and the maximum principle are equivalent. As a result, the MLP limiting is less sensitive to local mesh distribution and faithfully represents multi-dimensional flow physics.



(a) (b) (c)

Figure 1. Comparison of stencils involved in limiting and the maximum principle. Shaded region is the stencil for the maximum principle, and dotted line is for limiting: (a) LCD, MLG limiter (b) Barth's limiter (c) MLP limiting.

NUMERICAL RESULTS

Linear Wave Problem

In order to examine accuracy of convection data, the present scheme is applied on linear wave problem. The wave velocity **a** is (1, 1), and the initial condition is double sine function with periodic boundary condition. The triangular elements are created by dividing uniform square elements along the diagonal. Figure 2 shows the numerical solutions at t = 1, when the periodic wave return to the initial location. Comparing to the Barth's limiter, MLP-u1 limiter gives much better accuracy and less grid-dependency.

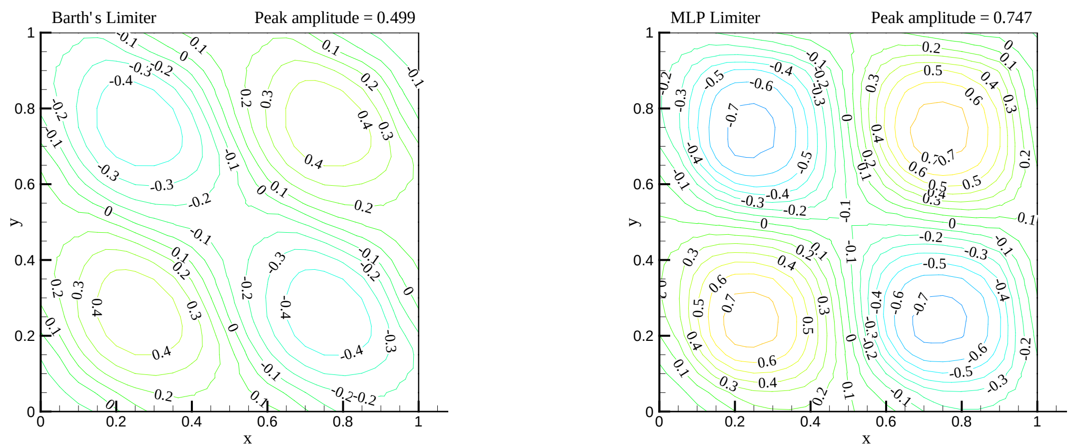


Figure 2. Comparison of numerical solutions: Barth's limiter (left) and MLP-u1 (right)

Interaction of Shock Wave with Cone

To compute complex three-dimensional flow, the interaction of a moving shock wave with a finite cone is considered. As the shock strikes the cone surface, a reflected shock is created and complex vortex structure is generated after the cone end.

Computational domain contains a half cylinder which covers the interval [-1.5, 3] in the x-direction with a half circle of $R=2.25$ in the y-z plane. The length of the half-circular cone is 1, tip radius is 0.02 and foot radius is 0.5. The tip of the cone is located at the origin. The initial condition of a moving shock with $M_s=1.3$ is imposed as follows.

$$\begin{aligned}(\rho_L, u_L, v_L, w_L, p_L) &= (2.122, 0.442, 0.0, 0.0, 1.805) \quad \text{if} \quad x < 0.1 \\ (\rho_R, u_R, v_R, w_R, p_R) &= (1.4, 0.0, 0.0, 0.0, 1.0) \quad \text{if} \quad x \geq 0.1.\end{aligned} \tag{5}$$

The number of mesh is 5.3 million tetrahedral elements. RoeM flux scheme and MLP-u1 limiter are applied. With PC cluster using 16 CPUs, MPI parallel computation was performed and it took 60 hours wall clock time to reach at $t= 2.5$.

Figure 3 shows the numerical schlieren images in the x-y plane and in the x-z plane. Compared to the experimental and numerical images, the flow structure looks very similar but the strength of the reflected shock becomes much weaker due to three-dimensional effect. This also confirms that the MLP-u1 limiter, combined with advanced numerical fluxes, guarantees sufficient resolution to capture complex shock-vortex structure.

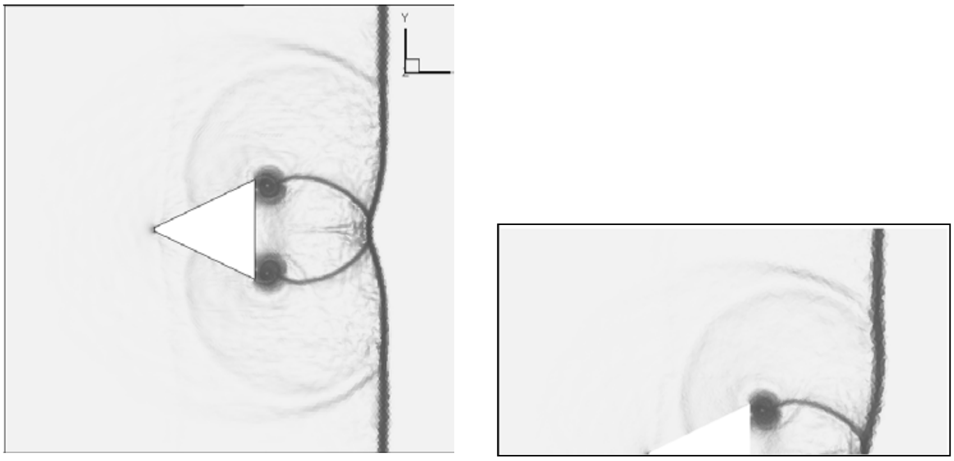


Figure 3. Numerical Schlieren images at $t=2.5$: x-y plane (left) and x-z plane (right)

CONCLUSION

A new multi-dimensional limiting process (MLP) on unstructured grids is developed with the multi-dimensional limiting condition whose basic idea is to control vertex values. The most distinguishable property of the MLP is to provide non-oscillatory problems in multi-dimensional flows. The satisfaction of maximum principle also guarantees the monotonicity of solution. Thanks to the property, the MLP can significantly increase accuracy, convergence/robustness and efficiency in multi-dimensional flows containing physical discontinuities. Various numerical results show the desirable characteristics of the proposed scheme, such as multi-dimensional monotonicity, improved accuracy and efficiency.

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